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TURBULENT MIXING NEAR THE GROUND FOR THE NESTED GRID MODEL

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Abstract

The physical principles that have been used to develop mixed-layer models for a surface layer of variable depth "h", are applied instead directly to the bottom layers of the Nested Grid Model (NGM). This eliminates the need to predict h explicitly and greatly simplifies the application of these ideas. The principles are reviewed in section 2. Generation of turbulence by shear at the top of the mixed region is ignored in this first formulation. In section 3 the principles are expressed as integrals with respect to the sigma coordinate of the NGM, and the NGM history variables of potential temperature and specific humidity are related to the "buoyancy". Section 4 spells out the details of the numerical method as applied to the discrete sigma layers of the model. The numerical method determines how many layers (K) participate in the mixing for each column. It then produces a uniform (larger) value of buoyancy in layers 1 through (K-1), and determines the appropriate decrease in the buoyancy of the capping layer K.

Section 5 shows how the changes in buoyancy derived in section 4 can be re-converted to changes in the NGM variables of potential temperature and specific humidity. An entrainment factor xm that describes the mixing between layers K-1 and K is used to effect this conversion. Section 6 describes how the mixing between layers K and K-1 could be modified when saturation occurs in layer K-1.

Tests with the Nested Grid Model in winter of 1985-6 were made in which only mechanical stirring was effective over land, because the model did not contain sensible heat flux over land, nor radiation. The stirring increased the horizontal averaged temperature in the bottom layer of the model over land by about 1 deg/day. Changes as large as 10 deg/day were observed in regions of strong wind and stable air.

The Appendix contains test results from computations with a single column using fixed values of surface heat flux and mechanical stirring. The functional dependence of the mixed layer depth with time agrees with well-known results from models that are formulated with an explicit mixed layer depth "h".

Turbulent mixing near the ground for the Nested Grid Model

1. Introduction.

Turbulence near the ground is affected by wind shear, surface roughness, and convection from the ground or ocean surface. (Turbulence generated by shear at the top of the mixed region will not be considered in the present paper.) The representation of these processes in redistributing heat and moisture in the Nested Grid Model (NGM) will follow the physical ideas of F. K. Ball (1960), J. Deardorff and G. Willis (1985), D. Lilly (1968), D. Randall (1984) and others, but with an important practical difference. These authors apply the physical ideas (first formulated in meteorology by Ball) to a model in which the bottom part of the atmosphere is occupied by a mixed layer of depth "h". Similar procedures have been used in oceanography to model the upper mixed layer, e.g. by Krause and Turner (1967). This depth must be forecast explicitly, together with the velocity, temperature, and moisture in the mixed layer. In the NGM, the physical ideas will be applied directly to the existing layer structure of the model; the explicit depth "h" will not be necessary.

2. The physical statements.

To express the physical ideas it is convenient to use the "anelastic" representation of the atmospheric equations (Ogura and Phillips, 1962). These are suitable for motions that are (a) slow enough that compressibility is important only for vertical displacements, but, (b) the motions can extend over a deep enough layer (several kilometers) that the upward decrease of density as a measure of inertia cannot be ignored. In this representation the motion is described as a deviation from a resting atmosphere of uniform potential temperature, θ . A subscript "a" will designate the variable in this reference atmosphere.

$$\theta_a = \Theta \quad (= \text{constant}) , \tag{2.1}$$

$$\pi_a = (p_a / 100 \text{ cb})^{R/c_p} = \pi_{oo} \left[1 - (g/c_p \Theta) z \right] , \tag{2.2}$$

$$\rho_a = \rho_{oo} \left[1 - (g/c_p \Theta) z \right]^{c_v/R} , \tag{2.3}$$

$$T_a = T_{oo} - (g/c_p) z = \pi_a \Theta , \tag{2.4}$$

$$\rho_{oo} = p_{oo} / R T_{oo} , \tag{2.5}$$

$$T_{oo} = \pi_{oo} \Theta . \tag{2.6}$$

p_{oo}, ρ_{oo}, and T_{oo} are the values at the ground, and "z" measures the height above the ground.

The anelastic system is as follows.

<u>The equation of motion:</u>

$$d\,\underset{\sim}{v} / dt = - \nabla_3 P + b \hat{\underset{\sim}{k}} + \underset{\sim}{F} . \tag{2.7}$$

(We can ignore the Coriolis force in deriving the turbulent effects.)

<u>The equation of continuity:</u>

$$\partial \rho_a w / \partial z + \nabla_2 \cdot \rho_a \underset{\sim}{V} = \nabla_3 \cdot \rho_a \underset{\sim}{v} = 0 . \tag{2.8}$$

<u>The first law of thermodynamics:</u>

$$d\, b /dt = \left(\frac{g}{c_p \Pi_a \Theta} \right) q . \tag{2.9}$$

"b" is the "buoyancy":

$$b = g (\theta - \Theta) / \Theta , \tag{2.10}$$

$\underset{\sim}{F}$ is the <u>molecular</u> viscous force while q is the heating rate (e.g. kilojoules per ton per sec) due to <u>molecular</u> conduction, radiation, and release of latent heat. "P" is a pressure-like variable, of dimensions $length^2 / time^2$. $\underset{\sim}{V}$ is the horizontal part of the three dimensional velocity vector, $\underset{\sim}{v}$. The explicit use of potential temperature in (2.10) will be convenient for the NGM, where potential temperature (as opposed to temperature) is the forecast variable.

The equation for the kinetic energy , with $K = (1/2) \rho_a \underset{\sim}{v}^2$, is derived from (2.7) and (2.8):

$$\frac{\partial K}{\partial t} = - \nabla_3 \cdot \rho_a \underset{\sim}{v} (K + P) + \rho_a b w + \rho_a \underset{\sim}{v} \cdot \underset{\sim}{F} . \tag{2.11}$$

The equation for the potential energy is derived from (2.8) and (2.9):

$$\frac{\partial (- \rho_a b z)}{\partial t} = \nabla_3 \cdot \rho_a \underset{\sim}{v} b z - \rho_a b w - \frac{(g z \rho_a)}{(c_p \Pi_a \Theta)} q . \tag{2.12}$$

The term $\rho_a b w$ represents the transformation from potential to kinetic energy.

We now manipulate the equations in the usual way by using the Reynolds' averaging convention for turbulence together with the usual boundary layer assumption that horizontal derivatives of turbulent fluxes can be ignored.

The first relevant equation is one for the change of the mean buoyancy:

$$\frac{\partial(\rho_a\ \overline{b})}{\partial t} = LS_b - \frac{\partial(\rho_a\ \overline{w'b'})}{\partial z} \quad . \tag{2.13}$$

The term LS_b includes all non-turbulent terms, such as advection of the mean buoyancy by the mean wind, and the mean heating rate $\overline{q}$. A second equation is obtained from (2.12) for the mean potential energy:

$$\frac{\partial(-\rho_a\ \overline{b}\ z)}{\partial t} = LS_p + \frac{\partial(\rho_a\ z\ \overline{w'b'})}{\partial z} - \rho_a\ \overline{b'w'}\ , \tag{2.14}$$

where LS_p includes all non-turbulent terms.

It will also be necessary to consider the equation for the turbulent kinetic energy $\overline{K'}$:

$$\overline{K'} = (1/2)\ \rho_a\ \overline{\underset{\sim}{v}'^2} \quad . \tag{2.15}$$

It is

$$\frac{\partial \overline{K'}}{\partial t} = -\frac{\partial(\rho_a\ \overline{w'(P'+K')})}{\partial z} + \rho_a\ \overline{\underset{\sim}{v}'\cdot\underset{\sim}{F}'} + \underset{\sim}{\tau}\cdot\frac{\partial\overline{\underset{\sim}{V}}}{\partial z} + \rho_a\ \overline{w'b'}\ , \tag{2.16}$$

in which

$$\underset{\sim}{\tau} = -\rho_a\ \overline{w'\underset{\sim}{V}'} \tag{2.17}$$

is the turbulent stress exerted by the air above level z on the air below level z.

We now integrate each of the three equations from z = δ to z = L , where δ is a very small distance close to the "ground", and L is a fixed height above the turbulence, such that all turbulent fluxes can (or will!) be ignored at L. (We can also ignore from now on the two large-scale terms since they will be forecast by the non-turbulent part of the model.)

$$\frac{\partial}{\partial t}\int_{\delta}^{L}\rho_a\ \overline{b}\ dz = \rho_a\ \overline{w'b'}\Big|_{z=\delta}\quad , \tag{2.18}$$

$$\frac{\partial}{\partial t}\int_{\delta}^{L}-\rho_a\ \overline{b}\ z\ dz = -\rho_a\ \delta\ \overline{w'b'}\Big|_{z=\delta} - \int_{\delta}^{L}\rho_a\ \overline{b'w'}\ dz\ , \tag{2.19}$$

$$\frac{\partial}{\partial t}\int_{\delta}^{L} \overline{K'}\, dz = \rho_a \overline{w'(P'+K')}\Big|_{z=\delta} + \int_{\delta}^{L} \rho_a \overline{\underset{\sim}{v}'\cdot\underset{\sim}{F}'}\, dz + \int_{\delta}^{L} \underset{\sim}{\tau}\cdot\frac{\partial \overline{\underset{\sim}{V}}}{\partial z}\, dz + \int_{\delta}^{L} \rho_a \overline{b'w'}\, dz \quad . \tag{2.20}$$

In (2.18) the rhs cannot be set equal to zero because while w' will tend to zero at the (level?) ground, b' will tend to increase because of the local hot spots associated with convection. The limit as δ goes to zero is that $\overline{w'b'}$ is the upward turbulent flux of buoyancy from the surface, denoted by B :

$$\rho_a \overline{w'b'} \text{ (at } z = \delta \text{)} = B \quad . \tag{2.21}$$

(B will be proportional to a linear combination of the heat flux H and the evaporation E, as discussed later in section 3.)

Our first physical statement about the effect of turbulence is the following integral:

$$\boxed{\frac{\partial}{\partial t}\int \rho_a \overline{b}\, dz = B} \tag{2.22}$$

(The integral extends from the ground to above the mixed region.)

To arrive at the second physical statement it is necessary to consider both (2.19) and (2.20):

In (2.20) we assume that

(a) the left side is close to zero--i.e. that there is an approximate balance between the terms on the right side, and (2.23)

(b) the term $\overline{w'(P'+K')}$ is negligible at the ground. (2.24)

In (2.19) we assume that

(c) the appearance of δ as a factor multiplying $\overline{w'b'}$ at $z = \delta$ allows us to set that term equal to zero. (Compare with (2.18), which had no δ multiplying it.) (2.25)

At this point we are left with a simple equation for the rate of change of the mean potential energy;

$$\frac{\partial}{\partial t} \int -\rho_a \overline{b} \, z \, dz = - \int \rho_a \overline{b'w'} \, dz , \qquad (2.26)$$

and a balance between three turbulent energy processes:

$$0 = \underbrace{\int \rho_a \overline{\underset{\sim}{v}' \cdot \underset{\sim}{F}'} \, dz}_{(I)} + \underbrace{\int \underset{\sim}{\tau} \cdot \frac{\partial \overline{\underset{\sim}{V}}}{\partial z} \, dz}_{(II)} + \underbrace{\int \rho_a \overline{b'w'} \, dz}_{(III)} . \qquad (2.27)$$

The three integrals have the following meaning.

(I) Viscous dissipation. This term is always negative.

(II) A change due to the turbulent Reynolds stress. This is normally positive in the boundary layer.

(III) A change due to upward flux of buoyancy. Its sign requires some discussion.

Consider first the "convective" case where heating (i.e. B) dominates and there is little mean wind. (III) must be positive since (I) is negative. Measurements by Deardorff and Willis (1985; see their figure 2) in laboratory convection experiments show considerable regularity in that the vertical integral of $\overline{b'w'}$ is proportional to B. In our notation the proportionality measured by them may be expressed as

$$\left(\int \rho_a \overline{b'w'} \, dz \right)_{CONV} = 0.4 \, h \, B , \qquad (2.28)$$

where h is the depth of the mixed layer.

We now consider the shear term, (II). Ball (1960) postulated that under conditions of strong wind (ignoring B) , it was not likely that (I) could balance all of (II) by itself. This would require that part of (II) be balanced by a <u>negative</u> value of (III):

$$\left(\int \rho_a \overline{b'w'} \, dz \right)_{SHEAR} = - \text{constant} \times \int \underset{\sim}{\tau} \cdot \frac{\partial \overline{\underset{\sim}{V}}}{\partial z} \, dz , \qquad (2.29)$$

where the constant is positive and less than one. Kato and Phillips (1969) performed laboratory experiments in which stably stratified water was agitated mechanically at the top. They concluded (in our notation) that

$$\left(\int \rho_a \, \overline{b'w'} \, dz \right)_{\text{SHEAR}} = -1.25 \, \rho_o \, u_*^3 , \qquad (2.30)$$

where u_* is the "friction velocity" :

$$\rho_o \, u_*^2 = | \underset{\sim}{\tau} |_{z=0} \quad . \qquad (2.31)$$

Denman and Miyake (1973) found that (2.30) gave reasonably accurate results when applied to the changes in the upper mixed layer of the ocean at ship PAPA, as induced by wind stirring. In terms of (2.26), this mixing leads to an <u>increase</u> in the potential energy-- a lifting of the center of mass-- by the mechanical stirring. Dreidonks (1981) performed calculations with a mixing model based on the above physical principles (but using an explicit mixed depth "h") to interpret turbulent field observations in Holland. He quotes the larger value of 2.5 derived by Kantha et al. (1977) as giving satisfactory results in his calculations. The latter value will be used therefore in the NGM instead of the Kato-Phillips value of 1.25 .

Our second equation is obtained by using both (2.28) and (2.30) to express the term on the right side of (2.26):

$$\boxed{\frac{\partial}{\partial t} \int - \rho_a \, \overline{b} \, z \, dz = -0.4 \, h \, B + 2.5 \, \rho_o \, u_*^3} \qquad (2.32)$$

One remark can already be made about the relative importance of the buoyancy flux B and the mechanical stirring. The depth over which b will be changing is "h" . The stress term in (2.32), however, is not proportional to h, whereas the other two terms in that equation are. We therefore can expect that the mechanical stirring will be most important only for small depths of the mixed layer, and to lose its importance as h increases. (Ball, in fact, considered the 3-km deep mixed layer occurring over the interior of Australia in the daytime. This value of h was large enough to lead him to ignore mechanical mixing in favor of that from convection. Wind stirring can however be expected to be more important in winter when B will be small over land and strong winds can produce a large u_*.)

The factor 0.4 in the results of Deardorff and Willis is less than 0.5. This is because there are negative values of $\overline{b'w'}$ around the top of the mixed region. This represents the <u>entrainment</u> of lighter fluid from the quiescent <u>region above the mixed layer into the turbulent</u> mixed layer.

3. Relation between fluxes of buoyancy, heat, and moisture, and the NGM "history variables".

Before specifying how (2.22) and (2.32) will be used in the NGM, it is necessary to correct the buoyancy formula for the effect of virtual temperature. The normal definition of virtual potential temperature is to multiply θ with the factor $(1 + 0.609\ q)$, where q is the specific humidity. This is a non-linear relation. The mixing process will determine changes in buoyancy, not θ or q. The former changes must, at the end, be converted into changes of θ and q. A linear relation for virtual potential temperature will simplify this conversion. To obtain this, we first note that in the mixed region for any one column, the variations of θ and q will be small. We therefore write

$$\theta_v = (1 + 0.609\ q_1)\ \theta + (0.609\ \theta_1)\ q \tag{3.1}$$

where θ_1 and q_1 are the values in the bottom layer of the model at each column. We consider this as in effect redefining the reference atmosphere on a local basis, so that θ_1 and q_1 in (3.1) can be considered as constants within each column.

The buoyancy flux B is then given by

$$\begin{aligned} B &= \overline{\rho}\ \overline{w'b'} = \overline{\rho}\ (g/\Theta)\ \overline{w'\theta_v'} \\ &= (g/\Theta)\left[(1+.609q_1)\ \overline{\rho}\ \overline{w'\theta'} + (0.609\ \theta_1)\ \overline{\rho}\ \overline{w'q'}\right] \\ &= (g/\Theta)\left[\left((1+.609q_1)/\pi_s c_p\right)\ HF + (0.609\ \theta_1)\ EV\right] \end{aligned} \tag{3.2}$$

where

HF = heat flux from the surface (kilojoules per square meter per second)

and (3.3)

EV = evaporation rate from the surface (tons of water per square meter per second)

The NGM has a sigma coordinate system in which sigma increases from zero at the earth's surface to one at zero pressure:

$$\sigma = (p_s - p)/p_s, \tag{3.4}$$

where p_s is the surface pressure. (The turbulence will not change p_s.) The "history" variables for temperature and specific humidity in the NGM are the products $p_s\ \theta$ and $p_s\ q$. (In the NGM code, p_s is stored in units of 100 cbs. The final statement in section 4 will describe how this convention for p_s can be easily accomodated.)

We now introduce several minor approximations into the integrals appearing in (2.22) and (2.32). These approximations will simplify greatly the arithmetic in CYBER vector operations, with little effect on the end result. Firstly we set

$$\rho_a \, dz = \rho dz = - dp / g = + p_s \, d\sigma / g \quad . \tag{3.5}$$

Secondly, we first replace z in (2.32) by z in the reference atmosphere,

$$z = (c_p \Theta / g) (1 - \Pi_a / \Pi_{oo}), \tag{3.6}$$

and then replace Π_a by Π from the NGM definition:

$$z = (c_p \Theta / g) (1 - \Pi / \Pi_{oo}). \tag{3.7}$$

The advantage of this will become apparent later. (This definition of z is of course not accurate enough to be used in computing the horizontal pressure force in the NGM forecast equations. But the purpose of z in (2.32) is to assign weights to the buoyancy tendency, as a function of elevation, that are different from those in (2.22), and (3.7) is more than adequate for this purpose.) The function Π in the NGM is given by:

$$\begin{aligned} \Pi &= (p / 100\text{cb})^{\kappa} = (p_s/100\text{cb})^{\kappa} (1 - \sigma)^{\kappa} \quad , \\ &= \Pi_{oo} (1 - \sigma)^{\kappa} \end{aligned} \tag{3.8}$$

where $\kappa = R / c_p$.

Thirdly, we recognize that the virtual temperature effects imply that the buoyancy "$\bar{b}$" must be defined as

$$\bar{b} = g (\theta_v - \Theta) / \Theta \quad , \tag{3.9}$$

where θ_v is the virtual temperature in the NGM as defined by (3.1). The second term, being constant, can be ignored. Equations (2.22) and (2.32) can now be written as

$$\int \partial (p_s \theta_v) / \partial t \; d\sigma = \Theta B \quad , \tag{3.10}$$

and

$$\int (1 - \Pi / \Pi_{oo}) \, \partial (p_s \theta_v) / \partial t \; d\sigma = (g/c_p) (0.4 \, h \, B - 2.5 \rho_s \, u_*^3) \tag{3.11}$$

The last of these may be combined with (3.10) and (3.8) to give

$$\int (1 - \sigma)^{\kappa} \, \partial (p_s \theta_v)/\partial t \; d\sigma = (\Theta - 0.4 \, g \, h / c_p) B + (2.5 \, g \, \rho_s / c_p) u_*^3 \tag{3.12}$$

The time integration procedure in the NGM involves a preliminary step from t to $t + (1/2)\,dt$, followed by a full time step from t to $t + dt$. All friction and turbulence terms are applied only in the full time step; this will also be true for for the present form of vertical mixing.

We should also recognize now that the goal of this process is to determine the changes in $p_s\,\theta$ and the changes in $p_s\,q$; changes in buoyancy (i.e. $p_s\,\theta_v$) are only an intermediate step. It will be convenient to use the following notation for these variables.

$$\alpha = p_s\,\theta \quad , \tag{3.13}$$

$$\gamma = p_s\,q \quad , \tag{3.14}$$

$$\beta = p_s\,\theta_v \quad . \tag{3.15}$$

From (3.1), we have that

$$\beta = (1 + .609q_1)\,\alpha \quad + \quad (0.609\,\theta_1)\,\gamma \quad . \tag{3.16}$$

In one time step the turbulence will produce the following changes in α and γ for a column.

$$\begin{aligned} \int \delta\alpha\, d\sigma &= g\,dt \frac{\partial}{\partial t} \int \rho\,\theta\,dz = g\,dt\,\rho_s\,(\overline{w'\theta'})_s \\ &= (g\,dt\,/\,\Pi_s\,c_p)\ \mathrm{HF} = H^* \,. \end{aligned} \tag{3.17}$$

$$\begin{aligned} \int \delta\gamma\, d\sigma &= g\,dt \frac{\partial}{\partial t} \int \rho\,q\,dz = g\,dt\,\rho_s\,(\overline{w'q'})_s \\ &= (g\,dt)\ \mathrm{EV} = E^* \,. \end{aligned} \tag{3.18}$$

For symmetry we also define a comparable quantity for the surface stress term.

$$W^* = dt\,(\rho_s\,u_*^3) \quad . \tag{3.19}$$

HF, EV, and ($\rho_s u_*^3$) will form the input of forcing terms for the mixing computations. The starred forms above are presented here only for notational convenience.

4. Efficient solution of the mixing equations.

Equations (3.10) and (3.12) will be applied to each column of grid points in the NGM by expressing the sigma integrals as a sum over the bottom sigma layers of the model. The layers involved are determined by the following definition.

> The mixing region will consist of layers $k = 1,2,3,--,K$, where layer K is the first layer (at time t) such that θ_v (or β) exceeds the value of θ_v (or β) in the mixed layers beneath it.

The integrals will therefore be replaced by sums over layers 1 through K . (A more precise definition of when β_K exceeds β in the layers underneath it will be given in (4.31) near the end of this section.)

As stated in this way, there would be K values of β (t+dt) to be solved for, from only two equations. This indeterminacy (for K greater than 2) is however only an apparent one because we view the mixing process in this time step as resulting in a uniform value of β (t+dt) in the layers $k = 1,2,--,(K-1)$, and a new value of β in layer K . Thus there are really only two unknowns.

Let $\delta\beta_K$ and $\delta\beta_m$ denote the changes in the buoyancy β for layer K and the layers 1 through (K-1) .

$$\delta\beta_K = \beta_K(t+dt) - \beta_K(t) , \tag{4.1}$$

$$\delta\beta_m = \beta_m(t+dt) - \beta_m(t) . \tag{4.2}$$

When K is greater than 2 , $\beta_m(t)$ will be defined as the following average value:

$$\beta_m(t) = \sum_{k=1}^{K-1} \Delta\sigma_k \, \beta_k(t) \;/\; S_K \tag{4.3}$$

where S_K is the value of sigma at the base of layer K:

$$S_K = \sum_{k=1}^{K-1} \Delta\sigma_k . \tag{4.4}$$

The finite sum counterpart of the integral statement (3.10) is now

$$S_K \, \delta\beta_m + \Delta\sigma_K \, \delta\beta_K = X, \tag{4.5}$$

where X is given by

$$X = (1 + 0.609\ q_1)\ H^* + (0.609\ \theta_1)\ E^* . \qquad (4.6)$$

To express the counterpart of (3.12) it is convenient first to introduce the notation

$$r = (1 - \sigma)^{\kappa} . \qquad (4.7)$$

The counterpart of (3.12) becomes

$$R_K\ \delta\beta_m + \Delta\sigma_K\ r_K\ \delta\beta_K = Y . \qquad (4.8)$$

R_K is equal to

$$R_K = \sum_{k=1}^{K-1} \Delta\sigma_k\ r_k , \qquad (4.9)$$

and Y is

$$Y = \left(1 - \frac{0.4\ g\ h}{c_p\ \Theta} \right) X + \left(\frac{2.5\ g}{c_p} \right) W^* . \qquad (4.10)$$

At this point we must define the Deardorff-Willis length "h". We will take it as equal to the vertical extent of layers 1 through K-1. A value for this that is consistent with (3.7) is

$$\frac{g\ h}{c_p\ \Theta} = (1 - r^*_K) , \qquad (4.11)$$

where

$$r^*_K = (1 - S_K)^{\kappa} \qquad (4.12)$$

(Note that this last step has removed all reference to Θ , the potential temperature of the reference atmosphere.) This yields a moderately simple simple expression for Y:

$$Y = \left[(1 - 0.4) + 0.4\ r^*_K \right] X + \frac{2.5\ g}{c_p} W^* . \qquad (4.13)$$

The solutions of (4.4) and (4.7) are

$$\delta\beta_K = \frac{R_K\ X - S_K\ Y}{\Delta\sigma_K\ (R_K - r_K\ S_K)} , \qquad (4.14)$$

and

$$\delta\beta_m = \frac{-\ r_K\ X + Y}{R_K - r_K\ S_K} . \qquad (4.15)$$

It is however worthwhile to expand these expressions for greater clarity. First we define a mean value of r by the ratio

$$\bar{r}_K = R_K / S_K$$
$$= \sum r_k \Delta\sigma_k \;/ \sum \Delta\sigma_k \quad . \qquad (4.16)$$

r decreases slowly with increasing sigma (or K), from a value of one at sigma = 0 to about 0.9 at sigma = 0.5 . However it satisfies the inequality

$$r_K < \bar{r}_K < 1 \quad . \qquad (4.17)$$

The denominators of (4.14) and (4.15) are therefore positive.

We now introduce the expression (4.13) for Y, and arrive at the following expressions for the changes in the buoyancy of layer K and the uniform buoyancy of the K-1 layers underneath layer K .

$$\Delta\sigma_K (\bar{r}_K - r_K) \; \delta\beta_K = $$
$$- \left[(1 - \bar{r}_K) - 0.4 \; (1 - r^*_K) \right] X - \left(\frac{2.5 \; g}{c_p} \right) W^* \quad , \qquad (4.18)$$

and

$$S_K (\bar{r}_K - r_K) \; \delta\beta_m = $$
$$\left[(1 - r_K) - 0.4 \; (1 - r^*_K) \right] X + \left(\frac{2.5 \; g}{c_p} \right) W^* \quad . \qquad (4.19)$$

The right sides of these two equations differ in only two regards: the outermost signs are different, and (4.18) uses $\bar{r}_K$ while (4.19) uses r_K. The square brackets multiplying X can be converted into the following expression by inserting the adiabatic "z" that is defined in (3.7).

$$[\quad] = \frac{g}{c_p \Theta} (z' - 0.4 \; z^*) \quad . \qquad (4.20)$$

z* is "z" at the top of layer K-1 . For (4.19), z' is "z" in the middle of layer K . The square bracket is therefore positive in (4.19). For (4.18), z' is equal to

$$z' \; (4.18) = \sum_{k=1}^{K-1} z_k \Delta\sigma_k \div \sum_{k=1}^{K-1} \Delta\sigma_k \quad . \qquad (4.20)$$

This is the mean "z" of layers 1 through K-1. This in turn is equal (approximately) to 0.5 z* . The square bracket will be positive also for (4.18) -- although smaller than that in (4.19). This arrangement of signs for the buoyancy changes is as it should be, with negative changes in the layer just above the completely mixed layers and positive changes in the completely mixed layers underneath.

A precise definition of the determination of layer K -- i.e., the first layer above the completely mixed region--- is now possible. To do this, we note that (4.18)-(4.19) can be rewritten as

$$\delta\beta_K = - AA_K \ (FX - BB_K \ X) \ , \tag{4.21}$$

and

$$\delta\beta_m = + CC_K \ (FX - DD_K \ X) \ , \tag{4.22}$$

FX has been defined as the following combination (independent of K):

$$FX = (1 - 0.4) X + (2.5 g / c_p) W^* \ , \tag{4.23}$$

and the following positive functions of K have been defined.

$$AA_K = 1 / (\Delta\sigma_K (\overline{r}_K - r_K)) \ , \tag{4.24}$$

$$BB_K = (\overline{r}_K - 0.4 \ r^*_K) \ , \tag{4.25}$$

$$CC_K = 1 / [S_K (\overline{r}_K - r_K)] \ , \tag{4.26}$$

$$DD_K = (r_K - 0.4 \ r^*_K) \ . \tag{4.27}$$

In order to suppress small irregularities in β we define, for each layer K , a mean value of β for the layers underneath it at time t :

$$\beta_m = \sum_{k=1}^{K-1} \Delta\sigma_k \beta_k / S_K \ . \tag{4.28}$$

Equations (4.21) and (4.22) then predict, for any layer that is under consideration as possibly being the correct layer K, that

$$\begin{aligned} (\beta_K - \beta_m)_{t+dt} &= (\beta_K - \beta_m)_t \\ &\quad - \left[(AA + CC)_K \ FX - (AA \ BB + CC \ DD)_K \ X \right] \\ &= (\beta_K - \beta_m)_t - E_K \ . \end{aligned} \tag{4.29}$$

where E_K is positive.

The mixing should involve enough layers that the left side of (4.29) is not negative. The criterion for determining layer K is therefore that it is the first layer that satisfies the criterion

$$(\beta_K - \beta_m)_t \geq E_K \ . \tag{4.30}$$

As noted earlier, the NGM code expresses p_s in units of 100 cbs rather than the centibars implied in the equations of this Office Note. Allowance for this is made by simply dividing the time increment dt by 100 when multiplying the input variables HF, EV, and $\rho_s u_*^3$. This converts $\delta\beta$ into the proper NGM units for $p_s\theta_v$, and will also take care of the mass exchange xm and $\delta p_s\theta$ and $\delta p_s q$ that are derived in the next section.

5. Changes in potential temperature and specific humidity.

We turn now to the question of constructing the changes in the NGM "history variables", namely potential temperature (actually $\alpha = p_s \theta$) and specific humidity (actually $\gamma = p_s q$). We picture the change $\delta\beta_K$ in β of the top layer to have been brought about by an interchange of xm units of mass between the top layer and the layers underneath it. That is to say--

$$\delta\beta_K = \text{mass added times } \beta_m(t) - \text{mass lost times } \beta_K(t)$$

$$= \text{xm times } (\beta_m(t) - \beta_K(t)) . \qquad (5.1)$$

(The mass added is equal to the mass lost .) This defines xm as

$$\text{xm} = \frac{\delta\beta_K}{\beta_m(t) - \beta_K(t)} . \qquad (5.2)$$

The denominator is negative, and the mixing equations, as discussed at the end of the last section, will produce a negative numerator. The criterion (4.30) insures also that xm does not exceed one.

We now picture the same process with respect to $\alpha = p_s \theta$. Knowing xm from (5.2), we first solve for α_K at t+dt.

$$\alpha_K (t+dt) = (1 - \text{xm}) \alpha_K(t) + \text{xm}\, \alpha_m(t) , \qquad (5.3)$$

where

$$\alpha_m(t) = \sum_{k=1}^{K-1} \Delta\sigma_k \ \alpha_k(t) \ / \ S_K . \qquad (5.4)$$

(3.17) states the conservation of heat. It can be written as

$$S_K \alpha_m(t+dt) + \Delta\sigma_K \alpha_K(t+dt) = S_K \alpha_m(t) + \Delta\sigma_K \alpha_K(t) + H^* . \qquad (5.5)$$

This can now be solved for $\alpha_m(t+dt)$.

For γ ($= p_s q$) we can follow the same procedure.

In conclusion, some consideration is necessary for the case where the flux of buoyancy at the surface is directed downward (as would occur when the near-surface temperature is warmer than the underlying surface). In the case of negative H* , for example , the appropriate procedure would seem to be to first subtract from $p_s \theta$ in layer 1 the amount implied by the downward flux of heat, and then to apply the mixing process described in this note <u>but with H* set equal to zero</u>. Similar reasoning would apply to a downward flux of moisture at the ground (negative E*).

Tests were made with this surface mixing method in the winter of 1985-6. At this time, the NGM did not contain radiation, so that neither sensible flux nor evaporation were computed over land. Only the mechanical stirring was effective over land, therefore. The tests showed that an average warming of about one degree per day in the bottom layer (over land) was caused by the stirring. Changes as large as +10 degrees per day were obtained upon occasion in regions of strong wind with very cold bottom-layer temperatures.

6. Saturation at the top of the mixed layers.

The mixing process described above can lead to saturation in the top layer (K-1) of the mixed layers, especially in cold air that moves out over warm water. In one experimental forecast containing an outbreak of cold air over the Gulf of Mexico, a shallow (K=2) mixed layer characterized the initial cold air over the southeastern United States. When this air moved out over the Gulf, the mixed layers rapidly extended up to include layer 3. This occurred in the presence of strong subsidence. This subsidence evidently acted to prevent the mixed layer from extending up higher, by producing relatively warm air in layer 4, with a pronounced inversion. The strong evaporation then produced saturation in layer 3, while the relative humidity in layer 4 was less than 10%. The saturation in layer 3 was realistic in that satellite pictures showed overcast stratocumulus in the cold air. However, the model forecast precipitation in all of the cold air over the Gulf---more than suggested by the few ship observations.

The computations described in the preceding sections allow for the buoyant effect of moisture through its effect on the virtual potential temperature, but they ignore the effect of the release of latent heat in saturated air. When the air in layer K-1 (the top of the mixed layers) is saturated, air parcels from that layer that participate in the mixing process with layer K, will move upward along a moist adiabat instead of along a dry adiabat. They will therefore arrive in layer K with a larger buoyancy than if they were unsaturated. <u>Thus, the buoyancy β_K of the capping layer K will not be as great relative to the mixed layers underneath.</u>

A crude estimate of the reduced relative buoyancy of layer K can be obtained by the calculation

$$\beta_K \text{ (adjusted)} = \beta_K \text{ (original)} - p_s \, \delta\theta \quad , \tag{6.1}$$

where $\delta\theta$ is the change in θ of a parcel lifted along the moist adiabat from layer K-1 to layer K. This is given by the expression

$$\delta\theta = (p_K - p_{K-1}) \times (d\theta / dp)_{m \ adiab} \, . \tag{6.2}$$

A sufficiently accurate expression for $d\theta / dp$ is

$$(d\theta / dp)_{m \text{ adiab}} = - \frac{\theta \; q \, \chi (1 - \tau)}{p \; (q + \epsilon \chi \tau^2)} , \tag{6.3}$$

where

$$\tau = c_p \; T \; / \epsilon \; L \; , \tag{6.4}$$

and

$$\epsilon = R(\text{dry air}) \; / \; R(\text{vapor}). \tag{6.5}$$

The effect of this on the the mixing factor xm defined in (5.2) is to replace $\beta_K(t)$ in the denominator by the adjusted value from (6.1):

$$xm(adj) = \frac{(-\delta \beta_K)}{\beta_K(t;adj) - \beta_m(t)} \tag{6.6}$$

This will increase xm from the unsaturated value, and increase the turbulent flux of moisture upwards into layer K, with a corresponding reduction in the moisture remaining in the mixed layers k=1,2,--(K-1). The increase in xm will also increase the warming of the mixed layers. Both of these changes will reduce the saturation originally present in layer K-1.

The adjustment to $\beta_K(t)$ could make the denominator of (6.6) small enough that layer K no longer satisfies condition (4.30). Some experimentation will probably be needed to see if it is necessary then to proceed to the next layer for K, or whether a cheaper expedient is possible.

Acknowledgments. My first ideas on this method of formulating the surface mixing process for the NGM ignored the buoyancy integral in the equation for turbulent kinetic energy when the surface buoyancy flux was positive. (In other words, I had ignored the experimental measurements by Deardorff and Willis!) Dr. Douglas Lilly drew my attention to this important omission. Dr. Paul Long helped considerably in the early formulation stages and in drawing my attention to the thesis by A. Driedonks.

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APPENDIX

Test solutions of (4.18)-(4.19), together with (5.3)-(5.5), have been made for a single column. The forcing values were constant in time. The surface heat flux (HF) was either zero, or had the value

$$HF = \rho_s \times C_d \times \text{velocity} \times c_p \times \text{temperature difference}$$

$$= (1.125\ 10^{-3}) \times (2.0\ 10^{-3}) \times (5\ \text{m/sec}) \times 1005 \times (2\ \text{degrees})$$

$$= 0.02261\ \text{kjoules per square meter per second.}$$

(C_d is a drag coefficient.) The mechanical stirring (W) was also either zero, or had the value

$$\rho_s\, u_*^3 = \rho_s\ (\ C_d^{1/2}\ \text{velocity}\ )^3$$

$$= \rho_s\ (\ (2.0\ 10^{-3})^{1/2} \times 10\ \text{m/sec}\ )^3$$

$$= 0.0001\ \text{tons per second}$$

Evaporation was set equal to zero in all cases.

The delta sigma values were based on a uniform layer depth of about 300 meters, similar to that in the bottom layers of the NGM. The surface pressure was 1013.25 mbs at a height of zero meters above sea-level. Pressures in the middle of the layers were 995.4, 960.6, 927.1, 894.7, 863.4,--- millibars. Specific humidity values of 9.82, 9.48, 9.15, 8.83, 8.52,--- grams per kilogram were assigned to the layers. Two distributions of potential temperature were treated, corresponding respectively to temperature lapse rates of -6.5 and +20.0 degrees per kilometer, each with a temperature of 288K at sea level. A time step (dt) of 10 minutes was used, and calculations were continued in each case until layers 1 through 4 were uniformly mixed. The accompanying tables show the values of $p_s\theta$ at the moments when each successive layer became completely mixed.

The time required to mix a given number of layers is larger with the more stable lapse rate. The non-zero values of HF and W were chosen fortuitously so that approximately the same time is required by HF and by W to mix layer two with layer one. However, from then on the stirring case (HF = 0) falls behind the heating case (W = 0). This is in accord with the statements made near the end of section 2.

The growth of the number of completely mixed layers with time is simple in these cases. Each table contains a value of the time required to mix a certain layer, divided by the time to mix layer two with layer one, and then raised to the power 1/2 for the heating case and 1/3 for the stirring case. The linear growth of these numbers verifies that these calculations reproduce the well-known results deduced from the conventional model with an explicit value of "h".

The values of xm , the mass exchange variable, varied in these calculations from a minimum of 0.0005 upon starting a new layer to a maximum of 0.08 at the end of layer 4 in the -6.5 deg/km case.

LAPSE RATE = −6.5 degrees per kilometer

	HF = 0.02261, W = 0				HF = 0, W = 10^{-4}		
time(hrs)	0	3.0	8 1/6	17 1/3	3 1/3	13	32 1/6
layer							
5	295.045						
4	294.103			293.652			292.587
3	293.150		292.862	293.652		292.153	292.587
2	292.182	292.056	292.862	293.652	291.694	292.153	292.587
1	291.253	292.056	292.862	293.652	291.294	292.153	292.587
	– – – – – (time / 3 hrs)$^{1/2}$ – – – –				– – (time / 3 1/3 hrs)$^{1/3}$ – –		
	0	1	1.65	2.40	1	1.57	2.13

LAPSE RATE = +20.0 degrees per kilometer

	HF = 0.02261, W = 0				HF = 0, W = 10^{-4}		
time(hrs)	0	33 1/6	94 5/6	186 1/6	35 1/2	139 5/6	346 1/6
layer							
5	332.847						
4	323.208			319.714			308.395
3	313.729		311.619	319.714		304.125	308.395
2	304.404	303.564	311.619	319.714	299.703	304.125	308.395
1	295.286	303.564	311.619	319.714	299.703	304.125	308.395
	– – – – – (time / 33 1/6 hrs)1/2 – –				– – –(time / 35 1/2 hrs)1/3 –		
	0	1	1.69	2.37	1	1.58	2.14